TAKE HOME BONUS ASSIGNMENT

Due Date: Thursday, April 19th, 2018.

Consider a country populated by two overlapping generations, young and old. We will denote the young people by y and old by o. Assume that each generation only lives for two-periods, and so in each period t there two types of individuals: young individuals born in period t and old individuals born in period $t-1$. A consumer's utility is given by:

$$u(c_t^y) + \beta u(c_{t+1}^o) \tag{1}$$

$$u(c) = \frac{c^{1-\eta}}{1-\eta} \tag{2}$$

where c_t^y is the consumption of the generation born in period t when young, and c_{t+1}^o is the consumption of the same generation when they are old in period $t+1$. Additionally, assume the following:

1- Nobody dies or migrates into or out of the country from the generation born in period t. In other words, we have:

$$n_t^y = n_{t+1}^o. \tag{3}$$

2- Each generation get bigger at a rate of γ. In other words, populations growth is equal to:

$$n_{t+1}^y = (1+\gamma)n_t^y. \tag{4}$$

3- Each generation's endowment gets bigger at a rate of g. That is, the growth rate of the endowment of each generation is equal to:

$$y_{t+1}^y = (1+g)y_t^y. \tag{5}$$

4- The endowment of each generation also increases when they are old, because of the returns to experience. That is, the endowment of the cohort born in t in period $t+1$ is:

$$y_{t+1}^o = (1+e)y_t^y. \tag{6}$$

5- Suppose that taxes are proportional to consumer's endowments:

$$\tau^y y_t^y \quad \& \quad \tau^o y_t^o. \tag{7}$$

Part I:

Question 1:

Define the aggregate output at time t, Y_t, in terms of the number of young workers n_t^y, the number of old workers n_t^o and the individual endowments y_t^y and y_t^o. Write Y_t as a function of total income of the young workers $n_t^y y_t^y$ born in period t (and parameters γ, e, g).

> $$Y_t = n_t^y y_t^y + n_t^o y_t^o$$
> $$Y_t = n_t^y y_t^y \left(1 + \frac{1+e}{(1+\gamma)(1+g)}\right)$$

Question 2:

Determine the growth rate of the country, $\frac{Y_{t+1}-Y_t}{Y_t}$, as a function of γ and g only. Then assuming that γ and g are very small, rewrite the expression as function of γ and g only (i.e. assume that $\gamma * g \approx 0$). Also briefly explain why the parameter e does not affect the aggregate growth of the country.

> $$\frac{Y_{t+1}}{Y_t} = \frac{n_{t+1}^y y_{t+1}^y}{n_t^y y_t^y} = (1+\gamma)(1+g)$$
>
> The growth rate is $\gamma + g$ as a first-order approximation. e represents growth over the life-cycle.

Question 3:

Write total taxes at time t (from the yound and old individuals) as a function of the total income of young workers $n_t^y y_t^y$ and the parameters γ, e, g, τ^y, τ^o. Then express total taxes as a function of aggregate output Y_t and the parameters γ, e, g, τ^y, τ^o.

$$T_t = n_t^y \tau^y y_t^y + n_t^o \tau^o y_t^o = n_t^y y_t^y \left(\tau^y + \frac{1+e}{(1+\gamma)(1+g)} \tau^o \right)$$

$$T_t = \frac{\tau^y + \frac{1+e}{(1+\gamma)(1+g)} \tau^o}{1 + \frac{1+e}{(1+\gamma)(1+g)}} Y_t$$

Question 4:

Define $\omega = (1 - \tau^y) + (1 - \tau^o)\frac{1+e}{1+r}$.

(a) Assume that individuals can borrow or lend to a bank that has access to international financial markets. The bank fixes an interest rate at r for private agents and accepts only to deal with young individuals, as old workers would die before repaying their debt. Write the inter-temporal budget constraint for each individual in terms of consumption c_t^y and c_{t+1}^o, the interest rate r, the endowment when young y_t^y, and the parameter ω.

(b) Can you argue under which conditions an increase in τ^y harms individuals more than an increase in τ^o? Explain in a sentence.

$$c_t^y + \frac{c_{t+1}^o}{1+r} = (1-\tau^y) y_t^y + (1-\tau^o) \frac{y_{t+1}^o}{1+r}$$

$$c_t^y + \frac{c_{t+1}^o}{1+r} = \left((1-\tau^y) + (1-\tau^o) \frac{1+e}{1+r} \right) y_t^y = \omega y_t^y$$

ωy_t^y is discounted wealth. An individual is worse off when ω is reduced. If $r > e$, then a tax in first period is more painful.

5) Define the following parameters:

$$\Omega^y = \frac{[\beta(1+r)]^{-1/\eta}}{[\beta(1+r)]^{-1/\eta} + \frac{1}{1+r}}, \qquad \Omega^o = \frac{1}{[\beta(1+r)]^{-1/\eta} + \frac{1}{1+r}}$$

Determine the optimal consumption paths (i.e c_t^y and c_{t+1}^o that maximize each individual's utility). We will keep the notations c_t^y and c_{t+1}^o. Write c_t^y and c_{t+1}^o as a function of Ω^y, Ω^o, ω and y_t^y.

$$\left(\frac{c_t^y}{c_{t+1}^o}\right)^\eta = \frac{1}{\beta(1+r)} \qquad c_t^y + \frac{c_{t+1}^o}{1+r} = \omega y_t^y$$

We find

$$c_{t+1}^o = \frac{1}{[\beta(1+r)]^{-1/\eta} + \frac{1}{1+r}} \omega y_t^y = \Omega^o \omega y_t^y$$
$$c_t^y = \frac{[\beta(1+r)]^{-1/\eta}}{[\beta(1+r)]^{-1/\eta} + \frac{1}{1+r}} \omega y_t^y = \Omega^y \omega y_t^y$$

Question 6:

(a) Denote s_t^y and s_{t+1}^o as an individual's savings when he is young at time t and old at time $t+1$. First, write the definitions of s_t^y and s_{t+1}^o. What is the relationship between s_t^y and s_{t+1}^o using only the budget constraint? Explain in one sentence.

(b) Write individual savings s_t^y while taking into account the optimal consumption path of an individual as a function of y_t^y. Under what condition are savings negative when young? You will write this condition in terms of Ω^y, τ^y, τ^o, e and r only. Explain how and why this condition depends on τ^y, on τ^o, on e and on r.

By defintions

$$s_t^y = (1-\tau^y)y_t^y - c_t^y$$
$$s_{t+1}^o = (1-\tau^o)y_{t+1}^o + r.s_t^y - c_{t+1}^o$$

Constraint implies $s_t^y = -s_{t+1}^o$. This is the equivalent of $CC_1 = -CC_2$ in the model seen in class.

$$s_t^y = (1-\tau^y - \Omega^y\omega)y_t^y$$

The condition for negative savings is (after substituting ω)

$$(1-\Omega^y)(1-\tau^y) - \Omega^y(1-\tau^o)\frac{1+e}{1+r} < 0$$

Question 7:

Define the aggregate consumption at time t, C_t, in terms of the number of young workers n_t^y, the number of old workers n_t^o and the individual consumptions c_t^y and c_t^o. Write aggregate consumption, C_t, as a function of $n_t^y y_t^y$ (and parameters Ω^y, Ω^o, γ, g and ω). Show that aggregate consumption C_t can be written as a function of aggregate output Y_t (and parameters Ω^y, Ω^o, γ, g, ω and e). Briefly explain why is it a relevant property (in one sentence)? Is it compatible with the representative agent assumption seen in class?

$$
\begin{aligned}
C_t &= n_t^y c_t^y + n_t^o c_t^o = n_t^y c_t^y + n_{t-1}^y c_t^o \\
C_t &= n_t^y \Omega^y \omega y_t^y + n_{t-1}^y \Omega^o \omega y_{t-1}^y \\
C_t &= \left(\Omega^y + \frac{\Omega^o}{(1+\gamma)(1+g)} \right) \omega n_t^y y_t^y \\
C_t &= \left(\Omega^y + \frac{\Omega^o}{(1+\gamma)(1+g)} \right) \frac{\omega}{1 + \frac{1+e}{(1+\gamma)(1+g)}} Y_t
\end{aligned}
$$

Question 8:

Determine aggregate private savings S_t as a function of $n_t^y y_t^y$ (and parameters Ω^y, ω, τ^y, γ and g), using your answer from Question 6. Under which conditions are aggregate savings negative? Deduce the link between the sign of individual savings in first period, s_t^y, and aggregate savings S_t.

$$
\begin{aligned}
S_t &= n_t^y s_t^y + n_t^o s_t^o = n_t^y s_t^y - n_{t-1}^y s_{t-1}^y \\
S_t &= (1 - \tau^y - \Omega^y \omega) n_t^y y_t^y - (1 - \tau^y - \Omega^y \omega) n_{t-1}^y y_{t-1}^y \\
S_t &= (1 - \tau^y - \Omega^y \omega) \left(1 - \frac{1}{(1+\gamma)(1+g)} \right) n_t^y y_t^y
\end{aligned}
$$

Part II:

Assume now that the government has public debt D_t. To finance this debt, it collects taxes T_t in each period, and it also spends G_t in each period. Also assume that the government's interest rate is r.

Question 1:

Write the dynamics of the public debt D_t (do not forget it depends on the interest rate r). Also write the dynamics of individuals' private debt denoted D_t^P. Combine both to obtain an expression for the foreign asset position of the country.

> $$D_{t+1} - D_t = G_t - T_t + rD_t$$
> $$D_{t+1}^P - D_t^P = -S_t + rD_t^P$$
> $$CA_t = D_{t+1} + D_{t+1}^P - D_t - D_t^P$$
>
> $$B_t = -D_t - D_t^P$$

Question 2:

Assume that tax rates can vary over time: i.e. τ_t^y and τ_t^o do not have to be constant. The definition of Ricardian Equivalence is the following:

Ricardian Equivalence is an economic hypothesis proposing that when making their consumption decisions, consumers internalize the government's budget constraint. This implies that, for a given level of government spending, the method of financing that spending does not have an effect on agents' consumption decisions. As a result government's method of financing its spending does not have an effect on consumers' demand.

Do we have Ricardian Equivalence in this model? Argue your position using an example of a decrease in taxes τ_t^y compensated by in increase of τ_{t+s}^y.

> Obvioulsy Ricardian Equivalence does not hold. Assume that, at period t, the government decides to decrease τ_t^y by one unit and to increase τ_{t+2}^y by $(1+r)^2$ units.

Then nothing changes for the government's budget. The cohort born at period t is better off and the cohort born at period $t+2$ is worse off. The individuals in both cohorts will modify their consumption.

Question 3:

Write the government's inter-temporal budget constraint at time 0 a function of D_0, G_t, τ_t^y, τ_t^o, y_t^y, y_t^o, n_t^y, and n_t^o for $t = 1..\infty$.

$$\sum_{t=1}^{\infty} \frac{1}{(1+r)^t} G_t = (1+r)D_0 + \sum_{t=1}^{\infty} \frac{1}{(1+r)^t} \left(n_t^y y_t^y \tau_t^y + n_t^o y_t^o \tau_t^o \right)$$